

A Variance Inequality for Glauber dynamics with Application to Low Temperature Ising Model

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A variance inequality for spin-flip systems is obtained using comparatively weaker knowledge of relaxation to equilibrium based on coupling estimates for single site disturbances. We obtain variance inequalities interpolating between the Poincaré inequality and the uniform variance inequality, and a general weak Poincaré inequality. For monotone dynamics the variance inequality can be obtained from decay of the autocorrelation of the spin at the origin, i.e., from that decay we conclude decay for general functions. This method is then applied to the low temperature Ising model, where the time-decay of the autocorrelation of the origin is extended to arbitrary quasi-local functions.

Keywords: Glauber dynamics, weak Poincaré inequality, relaxation to equilibrium, coupling

1 Introduction

Variance estimates and related inequalities have a long history in the study of interacting particle systems. Classical inequalities are the log-Sobolev inequality or Poincaré's inequality. A basic distinction between various types of estimates is whether they deal with the mixing structure in space, with respect to some measure, or in time, with respect to some dynamics. It is well-established that strong mixing properties in space imply strong mixing properties in time, and vice versa[6, 2]. Often this connection is made via tensorization arguments of the corresponding inequalities.

In [1] it is shown how a different method, disagreement percolation[8], can be used to obtain a Poincaré inequality. The idea used is to track how the influence of a

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single spin-flip possibly percolates through space, and then use subcriticality of the percolation to obtain results.

However the picture is a lot less clear when only weaker mixing properties hold. One of the few general tools available are weak Poincaré inequalities, which allow to translate a weaker type of spatial mixing to a form of mixing in time.

In this chapter, we approach the problem of mixing in another direction. We go from a restricted form of decay of correlations in time to general decay of correlations in time. The idea is to track the influence of a single spin-flip through time and space. In systems with weak mixing properties typically the influence of such a single flip is limited, but there is the possibility of a big influence, which leads to moment conditions on certain coupling times.

Given that an interacting particle system with nearest-neighbour Glauber dynamics satisfies those coupling conditions we obtain variance estimates for the ergodic measures as well as the relaxation of the dynamics. In the case of attractive dynamics, the coupling condition can be relaxed to a condition on the auto-correlation of the spin at the origin. Using the recent progress in [5] on the low-temperature Ising model we can extend the results to obtain quasi-polynomial relaxation to equilibrium of the Glauber dynamics.

2 Definitions and Notation

2.1 Setting

We consider the state space $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$. For a function $f : \Omega \rightarrow \mathbb{R}$, which is generally assumed to be bounded and measurable, define

$$\nabla_x f(\eta) := f(\eta^x) - f(\eta), \quad \eta \in \Omega, x \in \mathbb{Z}^d,$$

where η^x is the configuration η flipped at x , i.e., $\eta^x(x) = -\eta(x)$ and $\eta^x(y) = \eta(y)$ for $y \neq x$. We call f local if $\nabla_x f = 0$ for all but finitely many $x \in \mathbb{Z}^d$. In addition, we define a family of semi-norms for functions on Ω ,

$$\|f\|_p := \left(\sum_{x \in \mathbb{Z}^d} \sup_{\eta \in \Omega} (\nabla_x f(\eta))^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

A probability measure μ on the space Ω is called a Markov random field if the probability of observing a plus-spin(or minus-spin) given the spin of all other sites depends only the spin of the nearest neighbours. In terms of a random variable ξ on Ω that means

$$\mu(\xi(x) = +1 \mid \forall y \neq x : \xi(y) = \eta(y)) = \mu(\xi(x) = +1 \mid \forall y, |y - x| = 1 : \xi(y) = \eta(y))$$

for any $\eta \in \Omega$. With this fact in mind, define

$$\begin{aligned} c_+(x, \eta) &= \mu(\xi(x) = +1 \mid \forall y \neq x : \xi(y) = \eta(y)); \\ c_-(x, \eta) &= \mu(\xi(x) = -1 \mid \forall y \neq x : \xi(y) = \eta(y)) = 1 - c_+(x, \eta). \end{aligned}$$

The conditional probabilities are called translation invariant if $c_+(x, \eta) = c_+(0, \tau_x \eta)$ where $\tau_x \eta(y) = \eta(x + y)$.

A natural dynamics with respect to μ is the Glauber dynamics, where spins at site x flip individually according to some rates $c(x, \eta)$. Here we choose the heat-bath Glauber dynamics, where the flip rates are given by the conditional probabilities c_+, c_- :

$$c(x, \eta) := \begin{cases} c_+(x, \eta), & \eta(x) = -1; \\ c_-(x, \eta), & \eta(x) = +1. \end{cases}$$

The associated Markov process $(\eta_t)_{t \geq 0}$ is then defined via its generator L acting on the core of local functions,

$$Lf(\eta) = \sum_{x \in \mathbb{Z}^d} c(x, \eta) \nabla_x f(\eta).$$

Let $\mathbb{P}_\eta, \eta \in \Omega$, be the path measures on the space of cadlag trajectories and $S_t f(\eta) = \mathbb{E}_\eta f(\eta_t)$ the corresponding semi-group.

2.2 Poincaré and uniform variance inequalities

The Dirichlet form \mathcal{E} associated to L is given by

$$\mathcal{E}(f, f) = -2 \int f(\eta) Lf(\eta) \mu(d\eta) = \sum_{x \in \mathbb{Z}^d} \int c(x, \eta) (\nabla_x f)^2(\eta) \mu(d\eta).$$

A Poincaré inequality is said to hold if for some $K > 0$

$$\text{Var}_\mu(f) \leq K \mathcal{E}(f, f) = K \sum_{x \in \mathbb{Z}^d} \int c(x, \eta) (\nabla_x f)^2(\eta) \mu(d\eta) \quad (1)$$

hold for all $f \in L^2(\mu)$. The Poincaré inequality is equivalent to a spectral gap of the (self-adjointed) generator L in $L^2(\mu)$ and implies exponential relaxation of the semi-group in $L^2(\mu)$. Under the assumption that $\inf_{\eta \in \Omega} c(\eta, 0) > 0$ (1) is equivalent to

$$\text{Var}_\mu(f) \leq K' \sum_{x \in \mathbb{Z}^d} \int (\nabla_x f)^2(\eta) \mu(d\eta) = K' \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^2(\mu)}. \quad (2)$$

A much weaker inequality is the uniform variance inequality

$$\text{Var}_\mu(f) \leq K'' \|f\|_2^2 = K'' \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_\infty. \quad (3)$$

To the authors knowledge this inequality is not related to any form of relaxation of the semi-group.

2.3 Weak Poincaré inequality

When the Poincaré inequality does not hold ($K = K' = \infty$) but (3) is too weak because one still wants to obtain some information about the relaxation speed to equilibrium one can go to other inequalities. One is the so-called weak Poincaré inequality, usually formulated as

$$\text{Var}_\mu(f) \leq \alpha(r)\mathcal{E}(f, f) + r\Phi(f), \quad \mu(f) = 0, r > 0, \quad (4)$$

where $\Phi(\lambda f) = \lambda^2 \Phi(f)$, $\Phi(f) \in [0, \infty]$, and α is a function decreasing to 0. This implies the following relaxation to equilibrium:

$$\text{Var}_\mu(S_T f) \leq \xi(T) \left(\sup_{t \geq 0} \Phi(S_t f) + \text{Var}_\mu(f) \right)$$

with $\xi(T) = \inf\{r \geq 0 : -\frac{1}{2}\alpha(r) \log(r) \leq T\}$ (see [7]).

3 Results and discussion

3.1 Main results

Let $\widehat{\mathbb{P}}_{\eta, \xi}$ be the basic coupling (based on the graphical construction, see Section 4. See also for example [4]) between two copies of the dynamics starting from the configurations $\eta, \xi \in \Omega$. Set

$$\theta_t(\eta) = c(0, \eta) \widehat{\mathbb{P}}_{\eta^0, \eta}(\eta_t^1 \neq \eta_t^2), \quad t \geq 0. \quad (5)$$

For $p \in [1, \infty]$ define the function $D_p : [0, \infty[\rightarrow [0, \infty]$ as

$$D_p(T) = \int_T^\infty (t+1)^{2d+2} \|\theta_t\|_{L^q(\mu)} dt,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The function D_p is going to determine the relaxation speed of $S_t f$ for general functions. Note that by definition D_p is decreasing.

Theorem 3.1. *Let μ be a translation invariant Markov random field, and S_t the associated heat-bath semi-group. Fix $p \in [1, \infty]$ and assume $D_p(0) < \infty$. For all $f : \Omega \rightarrow \mathbb{R}$ with $\|f\|_2 < \infty$ the following inequality holds:*

$$\text{Var}_\mu(S_T f) \leq C_d D_p(T) \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^p(\mu)}. \quad (6)$$

Here C_d is a universal constant depending only on the dimension d .

Remark For $T = 0$ we obtain the variance inequality

$$\text{Var}_\mu(f) \leq D_p(0) \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^p(\mu)},$$

which interpolates between the Poincaré inequality ($p = 1$) and the uniform variance inequality ($p = \infty$).

Remark The factor $t^{2d+2} = t^{2(d+1)}$ appears for two reasons. First the volume of a space-time cone in \mathbb{Z}^d starting at the origin and growing as time progresses is of order t^{d+1} . Then an application of Cauchy-Schwarz's inequality produces the square.

The need to obtain the necessary estimate for Theorem 3.1 in terms of a specific coupling instead of some other measure of suitable decay of correlations in time can be an obstacle to applications. If the spin-system is attractive that obstacle can be avoided. By exploiting that attractive spin-systems are well-adapted to the coupling $\widehat{\mathbb{P}}$ we reduce the dependence on the coupling to the auto-correlation of the origin:

Theorem 3.2. *Assume that the spin-system is attractive. Let $\phi(t) := \text{Var}_\mu(S_t g)$, $g(\eta) = \eta(0)$, be the auto-correlation of the spin at the origin. Then the function D_p can be estimated by*

$$D_p(T) \leq C'_d \int_T^\infty (t+1)^{3d+2} (\phi(t))^{\frac{p-1}{4p}} dt,$$

with a dimension dependent constant $C'_d > 0$.

A good example where this result can be applied is the two-dimensional low-temperature Ising model. Recently in [5] the estimate

$$\text{Var}_{\mu^+}(S_t g) \leq \exp\left(-e^{c(\beta)\sqrt{\log(t+1)}}\right)$$

was obtained, with $c(\beta)$ some temperature dependent constant. Combining this with Theorem 3.2 gives a variance estimate for general functions.

Corollary 3.3. *Fix $p > 1$. Let $\widetilde{D}_p : [0, \infty[\rightarrow [0, \infty[$ be given by*

$$\widetilde{D}_p(T) = c_p(\beta) \int_T^\infty \exp\left(8 \log(t+1) - \frac{p-1}{4p} e^{c(\beta)\sqrt{\log(t+1)}}\right) dt.$$

For all $f : \Omega \rightarrow \mathbb{R}$ the relaxation of the semi-group in the plus-phase is estimated by,

$$\text{Var}_{\mu^+}(S_T f) \leq \widetilde{D}_p(T) \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^p(\mu)}.$$

3.2 Discussion

If $D_1(0) < \infty$ Theorem 3.1 implies a Poincaré inequality, and hence exponentially fast convergence to equilibrium. If $\|\theta_t\|_{L^q(\mu)}$ decays exponentially fast Theorem 3.1 still implies exponentially fast decay of the variance, but with respect to a stronger norm. This, however, is sufficient to prove a spectral gap of the generator L or, equivalently, a Poincaré inequality.

Proposition 3.4. *Suppose $\int_0^\infty \|\theta_t\|_{L^q(\mu)} e^{\lambda t} dt < \infty$ for some $\lambda > 0$ and $1 \leq q \leq \infty$. Then $]-\lambda/2, 0[$ belongs to the resolvent set of L .*

In fact, we can say even more about the connection between $\|\theta_t\|_{L^q(\mu)}$ and the Poincaré inequality.

Proposition 3.5. *Suppose the spin system is attractive and $\inf_{\eta \in \Omega} c(\eta, 0) > 0$. If the spin system satisfies the Poincaré inequality, then $\|\theta_t\|_{L^q(\mu)}$ decays exponentially fast for any $1 \leq q < \infty$.*

This shows that for attractive spin systems equivalence between exponential decay of $\|\theta_t\|_{L^q(\mu)}$, $1 \leq q < \infty$, and the existence of a spectral gap.

It is then natural to ask if even stronger inequalities are implied by fast decay of $\|\theta_t\|_{L^\infty(\mu)}$. This is indeed the case:

Proposition 3.6. *Set $\widehat{D}(T) := \int_T^\infty (t+1)^{d+1} \|\theta_t\|_{L^\infty(\mu)}^{\frac{1}{2}} dt$ and suppose $\widehat{D}(0) < \infty$. Then for all $f : \Omega \rightarrow \mathbb{R}$ with $\|f\|_1 < \infty$ and all $T \geq 0$*

$$\|S_T f - \mu(f)\|_\infty \leq C_d^{\frac{1}{2}} \widehat{D}(T) \|f\|_1.$$

If $D_1(0) = \infty$, but $D_p(0) < \infty$ for some $p > 1$ (but with sub-exponential decay), it is natural to compare Theorem 3.1 with a weak Poincaré inequality. Under essentially the same conditions, we can prove the following weak Poincaré inequality.

Proposition 3.7. *Assume the conditions of Theorem 3.1 and $c(\eta, x) \geq \delta > 0$. Then for all $f : \Omega \rightarrow \mathbb{R}$ with $\sum_{x \in \mathbb{Z}} \|(\nabla_x f)^2\|_{L^p(\mu)}$ and all $t \geq 0$,*

$$\text{Var}_\mu(S_t f) \leq C_d \delta^{-1} D_1(0, R) \mathcal{E}(f, f) + D_p(R) \Phi_R(f),$$

where

$$\begin{aligned} D_p(0, R) &= \int_0^R (t+1)^{2d+2} \|\theta_t\|_{L^p(\mu)}, \\ \Phi_R(f) &= \frac{\text{Var}_\mu(S_R f)}{D_p(R)} \leq C_d \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^p(\mu)}. \end{aligned}$$

This weak Poincaré inequality leads (with a minor modification of the proof in [7]) to

$$\text{Var}_\mu(S_T f) \leq \xi(T) \left(C_d \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^p(\mu)} + \text{Var}_\mu(f) \right).$$

The decay $\xi(T)$ is of order $D_p(T^{\frac{1}{2d+3}})$, which is worse than the one from Theorem 3.1. The reason for that is that in the weak Poincaré inequality the diverging $D_1(0, R)$ is partially used, while in Theorem 3.1 only the converging $D_p(0, R)$ is used.

4 Graphical construction

The graphical construction of the Glauber heat bath dynamics is the encoding of the random evolution of the process η_t into basic random components and a deterministic function of this randomness and the initial configuration. It is a well-known tool in the study of spin and particle systems.

Let \overline{N} be a Poisson point process on $\mathbb{Z}^d \times [0, \infty[$ with intensity one (wrt. the counting measure on \mathbb{Z}^d and the Lebesgue measure on $[0, \infty[$). A point $(x, t) \in \overline{N}$ represents a *chance* of flipping the spin at site x and time t . To realize this chance let $\overline{U} = (\overline{U}_n)_{n \in \mathbb{N}}$ be a countable iid. collection of $[0, 1]$ -uniform random variables independent of \overline{N} . We assume that to each $(x, t) \in \overline{N}$ there is an associated U from \overline{U} (which can be realized by a bijection from \overline{N} to \mathbb{N} , and we simply write $\overline{U} : \overline{N} \rightarrow [0, 1]$). We denote the expectation with respect to \overline{N} and \overline{U} by $\int d\overline{N}$ and $\int d\overline{U}$.

The elementary step is then as follows. Given the configuration η_{t-} before a possible flip at $(x, t) \in \overline{N}$ and the to (x, t) associated random variable $U = \overline{U}((x, t))$ we determine the configuration η_t after the possible flip deterministically. All sites $y \in \mathbb{Z}^d, y \neq x$, are unchanged, i.e., $\eta_t(y) = \eta_{t-}(y)$. If $U < c_+(x, \eta_{t-})$, then $\eta_t(x) = +1$, otherwise $\eta_t(x) = -1$. Since we ignore the original spin at x and simply replace it with a new one drawn according to conditional probability given the other spins we call this a *resampling event*.

The configuration η_t is then given by the successive application of all resampling events to the initial configuration η_0 . As those are infinitely many steps one has to take care that this is indeed well-defined. The goal is to define a deterministic function Ψ which will output the configuration at time t , η_t , given the inputs $\overline{N}, \overline{U}$ and η_0 . We now focus on the precise construction of the graphical representation and its properties.

For a single resampling event the definition of Ψ is simple. Let $\Psi : \Omega \times (\mathbb{Z}^d \times [0, \infty[\times [0, 1]) \rightarrow \Omega$ be given by

$$\Psi(\eta, (x, t, u))(y) := \begin{cases} +1, & y = x, c_+(x, \eta) \leq u; \\ -1, & y = x, c_-(x, \eta) > u; \\ \eta(y), & y \neq x. \end{cases}$$

This definition is directly extended recursively to a finite number of resampling events. For $(x_n, t_n, u_n)_{1 \leq n \leq N} \subset \mathbb{Z}^d \times [0, \infty[\times [0, 1]$ with $t_1 < t_2 < \dots < t_N$,

$$\Psi(\eta, (x_n, t_n, u_n)_{1 \leq n \leq N}) := \Psi(\Psi(\eta, (x_1, t_1, u_1)), (x_n, t_n, u_n)_{2 \leq n \leq N}),$$

and $\Psi(\eta, \emptyset) = \eta$.

Definition 4.1. Let G be a countable subset of $\mathbb{Z}^d \times [0, \infty[$.

- a) A partial order $<_G$ on $\mathbb{Z}^d \times [0, \infty[$ is defined as follows: $(x, t) <_G (y, s)$ iff either $x = y$ and $t < s$ or there exists a finite subset $\{(x_1, t_1), \dots, (x_K, t_K)\} \subset G$ such that $t < t_1 < t_2 < \dots < t_K \leq s$ and $|x_m - x_{m-1}| = 1, 2 \leq m \leq K$, as well as $|x_1 - x| = 1$ and $x_K = y$.

- b) Write $T_x := \sup\{t : (x, t) \in G\}$, $x \in \mathbb{Z}^d$, and $G_{<x} := \{(y, t) \in G : (y, t) \leq_G (x, T_x)\}$. We call G locally finite, if $|G_{<x}| < \infty$ for all $x \in \mathbb{Z}^d$.
- c) For G^U a countable subset of $\mathbb{Z}^d \times [0, \infty[\times [0, 1]$ the definitions a) and b) are copied in the canonical way (projection of G^U onto $\mathbb{Z}^d \times [0, \infty[$).

The purpose of this definition becomes transparent by the following fact.

Lemma 4.2. For any $G^U \subset \mathbb{Z}^d \times [0, \infty[\times [0, 1]$ finite, $x \in \mathbb{Z}^d$ and $\eta \in \Omega$,

$$\Psi(\eta, G^U)(x) = \Psi(\eta, G_{<x}^U)(x).$$

Proof The nearest-neighbour property of c_+ means that to determine the new spin after a resampling event (x, t) it is sufficient to know the spin value of the neighbours of x . Those might depend on earlier resampling events, which have again nearest neighbour dependencies, and all resampling events (y, s) which have an influence on (x, t) satisfy $(y, s) <_G (x, t)$. \square

This leads to the final definition of Ψ . For G^U a locally finite subset of $\mathbb{Z}^d \times [0, \infty[\times [0, 1]$ (or $G \subset \mathbb{Z}^d \times [0, \infty[, U : G \rightarrow [0, 1], G^U := \{(x, t, U(x, t)) : (x, t) \in G\}$),

$$\Psi(\eta, G^U)(x) := \Psi(\eta, G_{<x}^U)(x), \quad x \in \mathbb{Z}^d.$$

An important property of the graphical construction evident here is that Ψ is tolerant to certain changes in the order of resampling events. Intuitively, a resampling event (x, t) is influenced only by resampling events which happen before t and are not too distant from x . This intuition can be formalized via the ordering $>_G$, which we now do.

Lemma 4.3. Let $G^U \subset \mathbb{Z}^d \times [0, \infty[\times [0, 1]$ be locally finite and $A, B \subset G^U$ a decomposition of G^U such that $\forall (x_1, t_1, u_1) \in A, (x_2, t_2, u_2) \in B : (x_1, t_1) \not>_G (x_2, t_2)$. In words, A does not happen after B . Then

$$\Psi(\eta, G^U) = \Psi(\Psi(\eta, A), B).$$

Proof Assume G^U is finite. If not, restrict to $G_{<x}^U$.

The proof is a consequence from the following basic fact. For $(x_i, t_i, u_i) \in \mathbb{Z}^d \times [0, \infty[\times [0, 1], i = 1, 2$, with $|x_1 - x_2| > 1$,

$$\Psi(\eta, \{(x_1, t_1, u_1), (x_2, t_2, u_2)\}) = \Psi(\Psi(\eta, (x_1, t_1, u_1)), (x_2, t_2, u_2)). \quad (7)$$

By the property of the decomposition for each $(x_1, t_1, u_1) \in A, (x_2, t_2, u_2) \in B$, either $t_1 < t_2$ or $|x_1 - x_2| > 1$. The proof of the lemma is an iterative application of fact (7). Let $a_i, i = 1..|A|$ be the elements of A ordered in increasing time. Starting from $\Psi(\eta, A \cup B) = \Psi(\Psi(\eta, \emptyset), \{a_i : i = 1, \dots, |A|\} \cup B)$, we can use fact (7) to move a_1 past all resampling events in B and perform this resampling event first:

$$\Psi(\eta, A \cup B) = \Psi(\Psi(\eta, \{a_1\}), \{a_i : i = 2, \dots, |A|\} \cup B).$$

Repeating this procedure for all other elements of A in their time-order then proves the claim of the lemma. \square

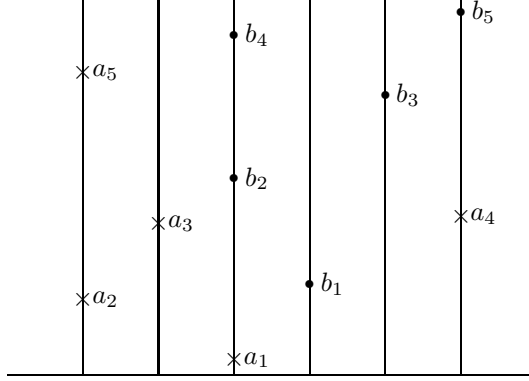


Figure 1: Resampling events a_1, \dots, a_5 do not depend on b_1, \dots, b_5 .

The final proposition of this section sums up the properties of the graphical representation.

Proposition 4.4. *Let $f : \Omega \rightarrow \mathbb{R}$ be quasi-local. The function Ψ has the following properties:*

- a) $\int \int f \left(\Psi(\eta, \overline{N}_t^{\overline{U}}) \right) d\overline{U} d\overline{N} = S_t f(\eta)$, where $\overline{N}_t^{\overline{U}} = \{(x, s, u) \in \overline{N}^{\overline{U}} : s \leq t\}$;
- b) For any locally finite $G \subset \mathbb{Z}^d \times [0, \infty[$, $\int \int f(\Psi(\eta, G^{\overline{U}})) d\overline{U} \mu(d\eta) = \int f(\eta) \mu(d\eta)$;
- c) For $\eta^1, \eta^2 \in \Omega$ the coupling $\widehat{\mathbb{P}}_{\eta^1, \eta^2}$ of \mathbb{P}_{η^1} and \mathbb{P}_{η^2} is defined via

$$\widehat{\mathbb{E}}_{\eta^1, \eta^2} f(\eta_t^1, \eta_t^2) = \int \int f \left(\Psi(\eta^1, \overline{N}_t^{\overline{U}}), \Psi(\eta^2, \overline{N}_t^{\overline{U}}) \right) d\overline{U} d\overline{N}.$$

Proof a) The point process $(\overline{N}_t^{\overline{U}})_{t \geq 0}$ is a Markov process on the subsets of $\mathbb{Z}^d \times [0, \infty[\times [0, 1]$ under $d\overline{U} d\overline{N}$ and with respect to the canonical filtration. The image process

$$\tilde{\eta}_t := \Psi(\eta, \overline{N}_t^{\overline{U}})$$

is also a Markov process since Ψ preserves the Markov property:

$$\tilde{\eta}_t = \Psi \left(\eta, \overline{N}_t^{\overline{U}} \right) = \Psi \left(\Psi(\eta, \overline{N}_s^{\overline{U}}), \overline{N}_t^{\overline{U}} \setminus \overline{N}_s^{\overline{U}} \right) = \Psi \left(\eta_s, \overline{N}_t^{\overline{U}} \setminus \overline{N}_s^{\overline{U}} \right), \quad t > s \geq 0.$$

The generator of $\tilde{\eta}_t$ is

$$\tilde{L}f(\eta) = \sum_{x \in \mathbb{Z}^d} \int_0^1 f(\Psi(\eta, (x, 0, u))) - f(\eta) du. \quad (8)$$

Since $\Psi(\eta, (x, 0, u))$ is either η or η^x , after integrating over u we obtain $\tilde{L}f = Lf$ on the core of local functions $f : \Omega \rightarrow \mathbb{R}$.

b) The proof follows the construction of Ψ . Let $G = \{(x, t)\}$ and write $\eta_+^x(x) = +1$, $\eta_+^x(y) = \eta(y)$ for $y \neq x$ (η_-^x analogue). Then

$$\begin{aligned} \int \int f(\Psi(\eta, G^{\bar{U}})) d\bar{U} \mu(d\eta) &= \int \int_0^1 f(\Psi(\eta, (x, t, u))) du \mu(d\eta) \\ &= \int c_+(x, \eta) f(\eta_+^x) + c_-(x, \eta) f(\eta_-^x) \mu(d\eta) \\ &= \int f(\eta) \mu(d\eta). \end{aligned}$$

For G a finite set the result is true by the iterative construction. For G countable but locally finite we observe that for local f only finitely many resampling steps have to be performed to determine the expectation of f .

c) By part a) $\widehat{\mathbb{E}}_{\eta^1, \eta^2} f(\eta_t^1) = S_t f(\eta^1)$ and $\widehat{\mathbb{E}}_{\eta^1, \eta^2} f(\eta_t^2) = S_t f(\eta^2)$, so $\widehat{\mathbb{P}}_{\eta^1, \eta^2}$ is indeed a coupling. \square

5 Proofs of the results

The first step is to rewrite the variance. As the following formula holds fairly generally and not just in this setting we formulate the lemma with more abstract conditions.

Lemma 5.1. *Let μ be an ergodic measure wrt. S_t and $f : \Omega \rightarrow \mathbb{R}$ such that $S_t f, (S_t f)^2 \in \text{dom}(L)$. Then, for $0 \leq T < S \leq \infty$,*

$$\text{Var}_\mu(S_T f) - \text{Var}_\mu(S_S f) = \int_T^S \int [L(S_t f - S_t f(\eta))^2](\eta) \mu(d\eta) dt \quad (9)$$

$$= \int_T^S \int \sum_{x \in \mathbb{Z}^d} c(x, \eta) (S_t f(\eta^x) - S_t f(\eta))^2 \mu(d\eta) dt. \quad (10)$$

Note that by ergodicity $\lim_{S \rightarrow \infty} \text{Var}_\mu(S_S f) = 0$.

Proof Since

$$\frac{d}{dt} \text{Var}_\mu(S_t f) = \int 2S_t f(\eta) L S_t f(\eta) \mu(d\eta),$$

we can express the variance as

$$\text{Var}_\mu(S_T f) - \text{Var}_\mu(S_S f) = \int_T^S \int -2S_t f(\eta) L S_t f(\eta) \mu(d\eta) dt.$$

By stationarity, $\int [L(S_t f)^2](\eta) \mu(d\eta) = 0$, hence

$$\begin{aligned} \text{Var}_\mu(S_T f) - \text{Var}_\mu(S_S f) &= \int_T^S \int [L(S_t f)^2](\eta) - 2S_t f(\eta) L S_t f(\eta) \mu(d\eta) dt \\ &= \int_T^S \int [L(S_t f - S_t f(\eta))^2](\eta) \mu(d\eta) dt. \end{aligned}$$

□

Note that $\|f\|_1 < \infty$ implies both $S_t f \in \text{dom}(L)$ and $(S_t f)^2 \in \text{dom}(L)$ in the setting of Glauber dynamics([4]).

The idea of the proof of Theorem 3.1 is to rewrite (9) using the graphical representation to describe the semi-group S_t . Then various applications of Hölder's inequality are used to separate different parts contributing to the variance formulation (9). However the calculation is fairly sensitive to the order in which different aspects are treated, and has one crucial non-trivial use of the graphical construction on the infinite volume.

We start by looking how the graphical construction can be used in light of Lemma 5.1. Let, by slight abuse of notation, $\overline{N} \subset \mathbb{Z}^d \times [0, \infty[$ be a fixed realization of the Poisson point process on $\mathbb{Z}^d \times [0, \infty[$, the set of resampling events. Almost surely this is a locally finite subset of $\mathbb{Z}^d \times]0, \infty[$. We denote all resampling events up to time t by $\overline{N}_t := \{(y, s) \in \overline{N} : s \leq t\}$.

To determine what influence a flip at site x has on the configuration at time t we use the graphical construction, particularly the partial order introduced in definition 4.1. Given the fixed realization \overline{N} , the cone

$$C_{t,x} := \{(y, s) \in \overline{N}_t : (y, s) >_{\overline{N}} (x, 0)\}$$

contains all resampling events which depend on the value of the initial configuration at site x , see also figure 2. Motivated by (10) we also introduce the same cone with another resampling event added at site x and time 0:

$$\tilde{C}_{t,x} := C_{t,x} \cup \{x, 0\}.$$

Given a realization of the independent uniform $[0, 1]$ variables associated to the resampling events, $\overline{U} : \overline{N} \rightarrow [0, 1]$, we extend the above sets to

$$\begin{aligned} \overline{N}_t^{\overline{U}} &:= \{(y, s, \overline{U}((y, s))) : (y, s) \in \overline{N}_t\}; \\ C_{t,x}^{\overline{U}} &:= \{(y, s, \overline{U}((y, s))) : (y, s) \in C_{t,x}\}. \end{aligned}$$

In the case of the added resampling event at $(x, 0)$ we assume a given $u \in [0, 1]$ to extend the event to $(x, 0, u)$. This leads to

$$\begin{aligned} \tilde{\overline{N}}_t^{\overline{U}} &:= \overline{N}_t^{\overline{U}} \cup \{(x, 0, u)\}, \\ \tilde{C}_{t,x}^{\overline{U}} &:= C_{t,x}^{\overline{U}} \cup \{(x, 0, u)\}, \end{aligned}$$

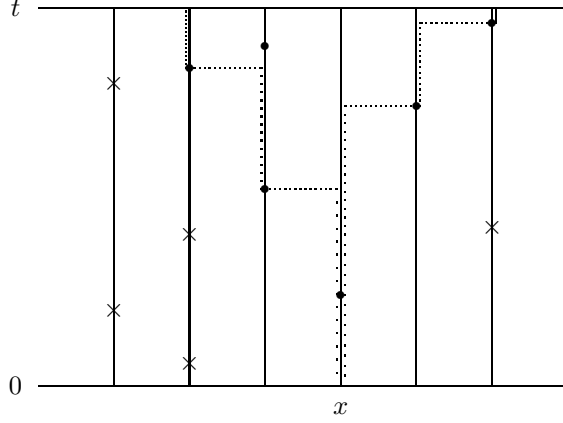


Figure 2: The cone $C_{t,x}$ containing all resampling events depending on $(0, x)$.

and, from $\eta \in \Omega$,

$$\tilde{\eta} := \Psi(\eta, (x, 0, u)).$$

Now we are ready to formulate the crucial idea. We want to compare the evolution of two configurations η_t^1, η_t^2 under the graphical construction coupling when started from two initial configurations $\eta, \tilde{\eta}$. By the graphical construction,

$$\begin{aligned} \eta_t^1 &= \Psi(\eta, \overline{N}_t^{\overline{U}}), \\ \eta_t^2 &= \Psi(\tilde{\eta}, \overline{N}_t^{\overline{U}}) = \Psi(\eta, \tilde{\overline{N}}_t^{\overline{U}}). \end{aligned}$$

By the reordering principle of the graphical construction in Lemma 4.3,

$$\begin{aligned} \eta_t^1 &= \Psi(\xi, C_{t,x}^{\overline{U}}), \\ \xi &= \Psi(\eta, \overline{N}_t^{\overline{U}} \setminus C_{t,x}^{\overline{U}}). \end{aligned} \tag{11}$$

Similarly,

$$\eta_t^2 = \Psi(\xi, \tilde{C}_{t,x}^{\overline{U}}). \tag{12}$$

So we can see ξ as a common ancestor of η_t^1 and η_t^2 in terms of the graphical construction (it is not an ancestor in time). This is very important, as both configurations only differ from ξ by a finite number of resampling events, namely those in $C_{t,x}^{\overline{U}}$ or $\tilde{C}_{t,x}^{\overline{U}}$ respectively. The proof of Theorem 3.1 is based on this observation, with Lemma 5.1 as a starting point.

To further facilitate the comparison of η_t^1, η_t^2 with ξ , write $C_{t,x}$ as the enumeration $\{(x_k, t_k, U_k), 1 \leq k \leq |C_{t,x}|\}$ with $t_k \geq t_{k-1}$ and $(x_0, t_0, U_0) = (x, 0, u)$. With this,

$$\xi_k := \Psi(\xi_{k-1}, (x_k, t_k, U_k)), \quad 1 \leq k \leq |C_{t,x}|, \quad (13)$$

$$\xi_0 := \xi,$$

$$\tilde{\xi}_k := \Psi(\tilde{\xi}_{k-1}, (x_k, t_k, U_k)), \quad 1 \leq k \leq |C_{t,x}|, \quad (14)$$

$$\tilde{\xi}_0 := \Psi(\xi, (x, 0, u)).$$

By Proposition 4.4 $\xi_k, \tilde{\xi}_k$ are μ -distributed since they are obtained via resampling steps. So we can describe η_t^1 and η_t^2 via finitely many flips from a common ancestor ξ , and each step in between is μ -distributed.

With the observations above we can rewrite part of (10) using the graphical representation.

Lemma 5.2. *Using above notation,*

$$\begin{aligned} & (S_t f(\tilde{\eta}) - S_t f(\eta))^2 \\ & \leq \widehat{\mathbb{P}}_{\tilde{\eta}, \eta}(\eta_t^1 \neq \eta_t^2) \int d\overline{N} (2|C_{t,x}| + 1) \int d\overline{U} \\ & \quad \left[\sum_{k=1}^{|C_{t,x}|} \left(\nabla_{x_k} f(\tilde{\xi}_{k-1}) \right)^2 + \sum_{k=1}^{|C_{t,x}|} \left(\nabla_{x_k} f(\xi_{k-1}) \right)^2 + \left(\nabla_x f(\xi) \right)^2 \right]. \end{aligned}$$

Proof Start with

$$\begin{aligned} (S_t(\tilde{\eta}) - S_t f(\eta))^2 &= \left(\widehat{\mathbb{E}}_{\tilde{\eta}, \eta}(f(\eta_t^1) - f(\eta_t^2)) \mathbb{1}_{\eta_t^1 \neq \eta_t^2} \right)^2 \\ &\leq \widehat{\mathbb{E}}_{\tilde{\eta}, \eta}(f(\eta_t^1) - f(\eta_t^2))^2 \widehat{\mathbb{P}}_{\tilde{\eta}, \eta}(\eta_t^1 \neq \eta_t^2). \end{aligned}$$

Now let $\widehat{\mathbb{P}}$ be the graphical construction coupling, then, in the notation of Section 4,

$$\widehat{\mathbb{E}}_{\tilde{\eta}, \eta}(f(\eta_t^1) - f(\eta_t^2))^2 = \int d\overline{N} \int d\overline{U} \left[f\left(\Psi(\tilde{\eta}, \overline{N}_t^{\overline{U}})\right) - f\left(\Psi(\eta, \overline{N}_t^{\overline{U}})\right) \right]^2.$$

Using (11) and (12),

$$\left[f\left(\Psi(\tilde{\eta}, \overline{N}_t^{\overline{U}})\right) - f\left(\Psi(\eta, \overline{N}_t^{\overline{U}})\right) \right]^2 = \left[f\left(\Psi(\xi, \tilde{C}_{t,x}^{\overline{U}})\right) - f\left(\Psi(\xi, C_{t,x}^{\overline{U}})\right) \right]^2. \quad (15)$$

This can be rewritten using the telescopic sum over the individual resampling steps (13),(14):

$$\begin{aligned} f\left(\Psi\left(\xi, C_{t,x}^{\overline{U}}\right)\right) - f(\xi_0) &= \sum_{k=1}^{|C_{t,x}|} f(\xi_k) - f(\xi_{k-1}), \\ f\left(\Psi\left(\xi, \tilde{C}_{t,x}^{\overline{U}}\right)\right) - f(\tilde{\xi}_0) &= \sum_{k=1}^{|C_{t,x}|} f(\tilde{\xi}_k) - f(\tilde{\xi}_{k-1}). \end{aligned}$$

Putting the telescopic sums into (15) and using the inequality $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$ leads to the upper bound

$$(2|C_{t,x}| + 1) \left[\sum_{k=1}^{|C_{t,x}|} \left(f(\tilde{\xi}_k) - f(\tilde{\xi}_{k-1}) \right)^2 + \sum_{k=1}^{|C_{t,x}|} (f(\xi_k) - f(\xi_{k-1}))^2 + (f(\tilde{\xi}_0) - f(\xi_0))^2 \right].$$

Notice that by construction, ξ_k and ξ_{k-1} are identical except for a possible flip at site x_k . Consequently, we can further estimate by

$$(2|C_{t,x}| + 1) \left[\sum_{k=1}^{|C_{t,x}|} (\nabla_{x_k} f(\tilde{\xi}_{k-1}))^2 + \sum_{k=1}^{|C_{t,x}|} (\nabla_{x_k} f(\xi_{k-1}))^2 + (\nabla_x f(\xi))^2 \right]. \quad \square$$

The next lemma deals with rearranging and separating integrals as well as condensing the individual terms as much as possible, continuing where Lemma 5.2 left off.

Lemma 5.3. For $1 \leq p \leq q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \int \mu(d\eta) \int_0^1 du (S_t f(\Psi(\eta, (x, 0, u))) - S_t f(\eta))^2 \\ & \leq \left(\int c(0, \eta) \theta_t(\eta)^q \mu(d\eta) \right)^{\frac{1}{q}} \left(\int (2|C_{t,0}| + 1)^2 d\bar{N} \right) \left(\sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^p(\mu)} \right), \end{aligned}$$

where

$$\theta_t(\eta) = \widehat{\mathbb{P}}_{\eta^0, \eta}(\eta_t^1 \neq \eta_t^2).$$

Proof We start by using Lemma 5.2 to estimate the inner term of

$$\sum_{x \in \mathbb{Z}^d} \int \mu(d\eta) \int_0^1 du (S_t f(\Psi(\eta, (x, 0, u))) - S_t f(\eta))^2.$$

Upon reordering some of the integrals and sums, we obtain

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \int d\bar{N} (2|C_{t,x}| + 1) \\ & \left[\sum_{k=1}^{|C_{t,x}|} \int \mu(d\eta) \int_0^1 du \int d\bar{U} \left(\nabla_{x_k} f(\tilde{\xi}_{k-1}) \right)^2 \widehat{\mathbb{P}}_{\tilde{\eta}, \eta}(\eta_t^1 \neq \eta_t^2) \right. \end{aligned} \quad (16)$$

$$+ \sum_{k=1}^{|C_{t,x}|} \int \mu(d\eta) \int_0^1 du \int d\bar{U} (\nabla_{x_k} f(\xi_{k-1}))^2 \widehat{\mathbb{P}}_{\tilde{\eta}, \eta}(\eta_t^1 \neq \eta_t^2) \quad (17)$$

$$\left. + \int \mu(d\eta) \int_0^1 du \int d\bar{U} (\nabla_x f(\xi))^2 \widehat{\mathbb{P}}_{\tilde{\eta}, \eta}(\eta_t^1 \neq \eta_t^2) \right]. \quad (18)$$

Now we use Hölder's inequality with respect to the integration $\int \mu(d\eta) \int_0^1 du \int d\bar{U}$. In all three cases this produces as second term

$$\left(\int \mu(d\eta) \int_0^1 du \int d\bar{U} \widehat{\mathbb{P}}_{\tilde{\eta}, \eta} (\eta_t^1 \neq \eta_t^2)^q \right)^{\frac{1}{q}}.$$

Note that, depending on u , $\tilde{\eta}$ is either η^x or η , in which case $\widehat{\mathbb{P}}_{\tilde{\eta}, \eta}(\eta_t^1 \neq \eta_t^2) = 0$. Using this as well as translation invariance shows that the above term equals

$$\left(\int c(0, \eta) \theta_t(\eta)^q \mu(d\eta) \right)^{\frac{1}{q}}.$$

The other term of Hölder's inequality varies slightly from line to line, but as it is mostly the same we focus on line (16):

$$\left(\int \mu(d\eta) \int_0^1 du \int d\bar{U} \left(\nabla_{x_k} f(\tilde{\xi}_{k-1}) \right)^{2p} \right)^{\frac{1}{p}}$$

Here we can finally use the fact that the configurations $\xi_k, \tilde{\xi}_k$ are μ -distributed. Because of this fact we have the following identity:

$$\begin{aligned} & \left(\int \mu(d\eta) \int_0^1 du \int d\bar{U} \left(\nabla_{x_k} f(\tilde{\xi}_{k-1}) \right)^{2p} \right)^{\frac{1}{p}} \\ &= \left(\int \mu(d\eta) (\nabla_{x_k} f(\eta))^{2p} \right)^{\frac{1}{p}} = \| (\nabla_{x_k} f)^2 \|_{L^p(\mu)}. \end{aligned}$$

Applying the same argument to (17) and (18),

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \int \mu(d\eta) \int_0^1 du (S_t f(\Psi(\eta, (x, 0, u))) - S_t f(\eta))^2 \\ & \leq \left(\int c(0, \eta) \theta_t(\eta)^q \mu(d\eta) \right)^{\frac{1}{q}} \\ & \quad \sum_{x \in \mathbb{Z}^d} \int d\bar{N} (2 |C_{t,x}| + 1) \left[2 \sum_{k=1}^{|C_{t,x}|} \| (\nabla_{x_k} f)^2 \|_{L^p(\mu)} + \| (\nabla_x f)^2 \|_{L^p(\mu)} \right]. \end{aligned}$$

By translation invariance of the law of \bar{N} ,

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^d} \int d\bar{N} (2 |C_{t,x}| + 1) \left[2 \sum_{k=1}^{|C_{t,x}|} \| (\nabla_{x_k} f)^2 \|_{L^p(\mu)} + \| (\nabla_x f)^2 \|_{L^p(\mu)} \right] \\ &= \sum_{x \in \mathbb{Z}^d} \int d\bar{N} (2 |C_{t,0}| + 1) \left[2 \sum_{k=1}^{|C_{t,0}|} \| (\nabla_{x_k+x} f)^2 \|_{L^p(\mu)} + \| (\nabla_x f)^2 \|_{L^p(\mu)} \right] \\ &= \int d\bar{N} (2 |C_{t,0}| + 1)^2 \sum_{x \in \mathbb{Z}^d} \| (\nabla_x f)^2 \|_{L^p(\mu)}. \end{aligned}$$

□

In order to proceed we need estimates on the size of $C_{t,0}$. The following two lemmas provides us with those.

Lemma 5.4. *Denote by $B_t \subset \mathbb{Z}^d$ the set of sites which are represented in $C_{t,0}$, i.e.,*

$$B_t := \{x \in \mathbb{Z}^d \mid \exists s \in [0, t] : (x, s) \in C_{t,0}\} \cup \{0\}.$$

Then there exist dimension-dependent constants $c_1, c_2 > 0$ such that

$$a) \int |B_t|^2 d\bar{N} \leq c_1(t+1)^{2d};$$

$$b) \sum_{x \in \mathbb{Z}^d} \left(\int \mathbb{1}_{x \in B_t} d\bar{N} \right)^{\frac{1}{2}} \leq c_2(t+1)^d.$$

Proof The proof rests on the observation that B_t is strongly related to first passage percolation: Consider first passage percolation with iid. exponentially distributed edge weights (see for example [3]). Let E be the edge set of \mathbb{Z}^d , and $r_e, e \in E$ iid. the exponentially distributed edge weights. Then the first passage percolation distance is $T(0, x) = \inf\{\sum_{e \in \gamma} r_e \mid \gamma \text{ path from } 0 \text{ to } x\}$. Now we compare the ball $\tilde{B}_t := \{x \in \mathbb{Z}^d : T(0, x) \leq t\}$ of reachable sites within distance t to B_t in terms of growth. Denote the outer boundary of a finite subset A of \mathbb{Z}^d by $\partial A = \{x \in \mathbb{Z}^d \setminus A \mid \exists y \in A : |x - y| = 1\}$. The rate at which a site $x \in \partial \tilde{B}_t$ is encompassed by \tilde{B}_t is given by the number of edges connecting x to \tilde{B}_t . On the other hand B_t grows to contain a site $x \in \partial B_t$ just at rate 1. Therefore \tilde{B}_t stochastically dominates B_t , and proving a) and b) for \tilde{B}_t suffices.

From the theory of first passage percolation (see [3], Theorems 3.10, 3.11) we use the following fact: There exist positive constants k_1, k_2, k_3 (possibly dimension-dependent) such that for all $x \in \mathbb{Z}^d$ with $|x| > k_1 t$:

$$\mathbb{P}(x \in \tilde{B}_t) = \mathbb{P}(T(0, x) \leq t) \leq k_2 e^{-k_3 |x|}. \quad (19)$$

To prove b),

$$\begin{aligned} \sum_{x \in \mathbb{Z}^d} \mathbb{P}(x \in \tilde{B}_t)^{\frac{1}{2}} &\leq \sum_{x: |x| \leq k_1 t} 1 + \sum_{x: |x| > k_1 t} k_2 e^{-k_3 |x|} \\ &\leq (2k_1 + 1)^d t^d + \sum_{x \in \mathbb{Z}^d} k_2 e^{-k_3 |x|} \\ &\leq c_2(t+1)^d \end{aligned}$$

for a suitable constant c_2 . To prove a), fix an integer $r > k_1 t$. Since $|\tilde{B}_t| > (2r+1)^d$ implies that at least one site in \tilde{B}_t lies outside a cube of size $2r+1$. Hence

$$\mathbb{P}\left(|\tilde{B}_t| > (2r+1)^d\right) \leq k_2 e^{-k_3(r+1)} 2d(2r+3)^{d-1},$$

which proves exponentially decaying tails for the volume of \tilde{B}_t . □

Utilizing Lemma 5.4 we now prove the second moment estimate of $|C_{t,0}|$ needed for Lemma 5.3.

Lemma 5.5. *There exists a dimension-dependent constant $C_d > 0$ so that the following estimate holds:*

$$\int (2|C_{t,0}| + 1)^2 d\bar{N} \leq C_d(t+1)^{2d+2}.$$

Proof Let B_t be as in Lemma 5.4. Then for each $x \in B_t$ we denote by t_x the time of first time of appearance of x in $C_{t,0}$,

$$t_x := \inf\{s \in [0, t] \mid (x, s) \in C_{t,0}\}.$$

We have

$$C_{t,0} = \bar{N} \cap \{(x, s) \in \mathbb{Z}^d \times [0, t] \mid x \in B_t, s \geq t_x\} \subset \bar{N} \cap \{(x, s) \in \mathbb{Z}^d \times [0, t] \mid x \in B_t\}.$$

Conditioned on B_t and t_x the last set is Poisson distributed with the addition of the points (x, t_x) , $x \in B_t$. Because of this, conditioned on B_t , $|C_{t,0}| - |B_t|$ is stochastically dominated by a Poisson distributed with parameter $t|B_t|$. As a consequence,

$$\int (2|C_{t,0}| + 1)^2 d\bar{N} \leq 4 \int (t+1)^2 (|B_t| + 1)^2 d\bar{N}.$$

Finally the estimate from Lemma 5.4,a) completes the proof. \square

With all ingredients present we can quickly prove the main result in form of a slightly more general lemma.

Lemma 5.6. *Let $f : \Omega \rightarrow \mathbb{R}$ with $\|f\|_2 < \infty$ and $0 \leq T \leq S$. Then*

$$\begin{aligned} & \text{Var}_\mu(S_T f) - \text{Var}_\mu(S_S f) \\ & \leq C_d \int_T^S (t+1)^{2d+2} \left(\int c(0, \eta) \theta_t(\eta)^q \mu(d\eta) \right)^{\frac{1}{q}} dt \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^p(\mu)}. \end{aligned}$$

C_d is a constant depending just on the dimension.

Proof Assume that f satisfies $\|f\|_1 < \infty$. This then implies that $S_t f, (S_t f)^2 \in \text{dom}(L)$ and by Lemma 5.1,

$$\text{Var}_\mu(S_T f) - \text{Var}_\mu(S_S f) = \int_T^S \int [L(S_t f - S_t f(\eta))^2](\eta) \mu(d\eta) dt.$$

By using the formulation of the generator using the graphical construction (see (8)),

$$Lf(\eta) = \sum_{x \in \mathbb{Z}^d} \int_0^1 f(\Psi(\eta, (x, 0, u))) - f(\eta) du,$$

we apply Lemma 5.3 and obtain

$$\text{Var}_\mu(S_T f) - \text{Var}_\mu(S_S f) \leq \int_T^S \int (2|C_{t,0}| + 1)^2 d\bar{N} \|\theta_t\|_{L^q(\mu)} dt \sum_{x \in \mathbb{Z}^d} \|(\nabla_x f)^2\|_{L^p(\mu)}.$$

Finally Lemma 5.5 gives us the estimate on $\int (2|C_{t,0}| + 1)^2 d\bar{N}$ to complete the proof.

If f only satisfies $\|f\|_2 < \infty$ we then approximate f by local functions. \square

Proof of Theorem 3.1 A direct consequence of Lemma 5.6 with $S = \infty$ and the estimate $(\int c(0, \eta) \theta_t(\eta)^q \mu(d\eta))^{\frac{1}{q}} \leq \|\theta_t\|_{L^q(\mu)}$. \square

We now prove Theorem 3.2, which is a modification of Theorem 3.1 for attractive spin-systems.

Proof of Theorem 3.2 This result is also based on Lemma 5.6. To estimate $(\int c(0, \eta) \theta_t(\eta)^q \mu(d\eta))^{\frac{1}{q}}$ in terms of the auto-correlation, we start with the fact that in the coupling the spread of discrepancies is limited to B_t (as in Lemma 5.4):

$$\theta_t(\eta) = \widehat{\mathbb{P}}_{\eta^0, \eta}(\eta_t^1 \neq \eta_t^2) \leq \widehat{\mathbb{E}}_{\eta^0, \eta} \sum_{x \in B_t} \mathbb{1}_{\eta_t^1(x) \neq \eta_t^2(x)} = \sum_{x \in \mathbb{Z}^d} \widehat{\mathbb{E}}_{\eta^0, \eta} \mathbb{1}_{x \in B_t} \mathbb{1}_{\eta_t^1(x) \neq \eta_t^2(x)}.$$

Next, since $\theta_t \leq 1$,

$$\begin{aligned} \int c(0, \eta) \theta_t(\eta)^q \mu(d\eta) &\leq \int c(0, \eta) \theta_t(\eta) \mu(d\eta) \\ &\leq \sum_{x \in \mathbb{Z}^d} \int \widehat{\mathbb{E}}_{\eta^0, \eta} \mathbb{1}_{x \in B_t} \mathbb{1}_{\eta_t^1(x) \neq \eta_t^2(x)} c(0, \eta) \mu(d\eta). \end{aligned}$$

We can now use Cauchy-Schwarz to obtain

$$\sum_{x \in \mathbb{Z}^d} \left(\int \mathbb{1}_{x \in B_t} d\bar{N} \right)^{\frac{1}{2}} \left(\int \widehat{\mathbb{E}}_{\eta^0, \eta} \mathbb{1}_{\eta_t^1(x) \neq \eta_t^2(x)} c(0, \eta)^2 \mu(d\eta) \right)^{\frac{1}{2}}. \quad (20)$$

Since the model is attractive the coupling $\widehat{\mathbb{P}}$ preserves an initial ordering. Since either $\eta^0 < \eta$ or $\eta < \eta^0$,

$$\begin{aligned} \widehat{\mathbb{E}}_{\eta^0, \eta} \mathbb{1}_{\eta_t^1(x) \neq \eta_t^2(x)} &= \frac{1}{2} \widehat{\mathbb{E}}_{\eta^0, \eta} |\eta_t^1(x) - \eta_t^2(x)| = \frac{1}{2} \left| \widehat{\mathbb{E}}_{\eta^0, \eta} (\eta_t^1(x) - \eta_t^2(x)) \right| \\ &= \frac{1}{2} \left| \mathbb{E}_{\eta^0} \eta_t(x) - \mathbb{E}_\eta \eta_t(x) \right|. \end{aligned}$$

When we use the notation $g_x(\eta) := \eta(x)$ and $m = \mu(g_0) = \mu(g_x)$,

$$\begin{aligned} \int \widehat{\mathbb{E}}_{\eta^0, \eta} \mathbb{1}_{\eta_t^1(x) \neq \eta_t^2(x)} c(0, \eta)^2 \mu(d\eta) &= \frac{1}{2} \int |S_t g_x(\eta^0) - S_t g_x(\eta)| c(0, \eta)^2 \mu(d\eta) \\ &\leq \frac{1}{2} \int |S_t g_x(\eta^0) - m| c(0, \eta)^2 \mu(d\eta) + \frac{1}{2} \int |S_t g_x(\eta) - m| c(0, \eta)^2 \mu(d\eta). \end{aligned}$$

Using $c(0, \eta) \leq 1$, as well as

$$\begin{aligned} \int |S_t g_x(\eta) - m| \mu(d\eta) &\leq \left(\int (S_t g_x(\eta) - m)^2 \mu(d\eta) \right)^{\frac{1}{2}} \\ &= \text{Var}_\mu(S_t g_x)^{\frac{1}{2}} = \text{Var}_\mu(S_t g_0)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \int |S_t g_x(\eta^0) - m| c(0, \eta) \mu(d\eta) &= \int |S_t g_x(\eta^0) - m| c(0, \eta) \frac{\mu(d\eta)}{\mu(d\eta^0)} \mu(d\eta^0) \\ &\leq \int |S_t g_x(\eta^0) - m| \mu(d\eta^0) \leq \text{Var}_\mu(S_t g_0)^{\frac{1}{2}} \end{aligned}$$

we estimate (20) and obtain

$$\int c(0, \eta) \theta_t(\eta)^q \mu(d\eta) \leq \sum_{x \in \mathbb{Z}^d} \left(\int \mathbb{1}_{x \in B_t} d\bar{N} \right)^{\frac{1}{2}} \text{Var}_\mu(S_t g_0)^{\frac{1}{4}}.$$

Furthermore Lemma 5.4 gives us an estimate for the sum, so that

$$\left(\int c(0, \eta) \theta_t(\eta)^q \mu(d\eta) \right)^{\frac{1}{q}} \leq c_2^{\frac{1}{q}} (t+1)^{\frac{d}{q}} \text{Var}_\mu(S_t g_0)^{\frac{1}{4q}}.$$

Omitting the q -th root where convenient and with a constant $C'_d = C_d(1 \vee c_2)$ as well as writing $\frac{1}{q} = \frac{p-1}{p}$ we obtain the result

$$D_p(T) \leq C'_d \int_T^\infty (t+1)^{3d+2} (\text{Var}_\mu(S_t g_0))^{\frac{p-1}{4p}} dt.$$

□

Proof of Proposition 3.4 Let $f : \Omega \rightarrow \mathbb{R}$ be a local function with $\mu(f) = 0$. By Theorem 3.1, for any $0 < \lambda' < \lambda$,

$$\|S_t f\|_{L^2(\mu)}^2 \leq \text{const} \cdot e^{-\lambda' t},$$

and for any $0 < a < \lambda'$

$$\int_0^\infty e^{at} \|S_t f\|_{L^2(\mu)}^2 dt < \infty. \quad (21)$$

Let $E_{f,f}$ be the associated measure wrt. to the spectral decomposition of $-L$. Then

$$\|S_t f\|_{L^2(\mu)}^2 = \int_0^\infty e^{-2\gamma t} E_{f,f}(d\gamma).$$

By (21),

$$\int_0^\infty \int_0^\infty e^{at-2\gamma t} E_{f,f}(d\gamma) dt < \infty.$$

Therefore $E_{f,f}([0, \lambda/2]) = 0$.

Let now $f \in L^2(\mu)$ and approximate it by local functions f_n . Assuming that $(f_n), f$ have norm 1 makes $E_{f_n, f_n}, E_{f,f}$ probability measures and E_{f_n, f_n} weakly converges to $E_{f,f}$. By the Portmanteau theorem $E_{f,f}([0, \lambda/2]) = 0$, which completes the proof. \square

Proof of Proposition 3.5 By the Poincaré inequality, the auto-correlation of the spin at the origin, $\phi(t) = \text{Var}_\mu(S_t g)$, $g(\eta) = \eta(0)$, decays exponentially fast. The proof of Theorem 3.2 contains the estimate of $(\int c(\eta, 0) \theta_t(\eta)^q \mu(d\eta))^{\frac{1}{q}}$ in terms of ϕ . \square

Proof of Proposition 3.6 We have

$$\|S_T - \mu(f)\|_\infty = \left\| \int_T^\infty L S_t f dt \right\| \leq \sup_{\eta \in \Omega} \int_T^\infty \sum_{x \in \Omega} |\nabla_x f(\eta)|.$$

Write $\delta_x(f) := \|\nabla_x f\|_\infty$. Then

$$\begin{aligned} \sum_{x \in \mathbb{Z}^d} |\nabla_x S_t f(\eta)| &\leq \sum_{x \in \mathbb{Z}^d} \widehat{\mathbb{E}}_{\eta^x, \eta} |f(\eta_t^1) - f(\eta_t^2)| \\ &\leq \sum_{x \in \mathbb{Z}^d} \widehat{\mathbb{E}}_{\eta^x, \eta} \left(\sum_{y \in \mathbb{Z}^d} \mathbb{1}_{\eta_t^1(y) \neq \eta_t^2(y)} \delta_y(f) \right) = \widehat{\mathbb{E}}_{\eta^0, \eta} \sum_{y \in \mathbb{Z}^d} \mathbb{1}_{\eta_t^1(y) \neq \eta_t^2(y)} \|f\|_1 \\ &\leq \widehat{\mathbb{E}}_{\eta^0, \eta} |C_{t,0}| \mathbb{1}_{\eta_t^1 \neq \eta_t^2} \|f\|_1 \end{aligned}$$

By the Cauchy-Schwarz inequality and Lemma 5.5,

$$\widehat{\mathbb{E}}_{\eta^0, \eta} |C_{t,0}| \mathbb{1}_{\eta_t^1 \neq \eta_t^2} \leq \widehat{\mathbb{P}}_{\eta^0, \eta}(\eta_t^1 \neq \eta_t^2)^{\frac{1}{2}} \left(\widehat{\mathbb{E}}_{\eta^0, \eta} |C_{t,0}|^2 \right)^{\frac{1}{2}} \leq \|\theta_t\|_{L^\infty(\mu)}^{\frac{1}{2}} C_d^{\frac{1}{2}} (t+1)^{d+1}.$$

Therefore

$$\|S_T - \mu(f)\|_\infty \leq \int_T^\infty C_d^{\frac{1}{2}} (t+1)^{d+1} \|\theta_t\|_{L^\infty(\mu)}^{\frac{1}{2}} dt \|f\|_1. \quad \square$$

Proof of Proposition 3.7 Fix $R > 0$. Then, by Lemma 5.6,

$$\begin{aligned} \text{Var}_\mu(S_t f) &\leq C_d D_1(0, R) \sum_{x \in \mathbb{Z}^d} \|(\nabla_x S_t f)^2\|_{L^1(\mu)} + \text{Var}_\mu(S_R S_t f) \\ &\leq C_d \delta^{-1} D_1(0, R) \mathcal{E}(S_t f, S_t f) + \text{Var}_\mu(S_R f) \\ &\leq C_d \delta^{-1} D_1(0, R) \mathcal{E}(f, f) + D_p(R) \Phi_R(f). \end{aligned}$$

The estimate $\Phi_R(S_t f) \leq C_d \sum_{x \in \mathbb{Z}^d} \|(\nabla_x S_t f)^2\|_{L^p(\mu)}$ is a direct consequence of Theorem 3.1. \square

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